Hunting for Division Algebras in Representations of Finite Groups

Def: A division algebra over a field k is f.d. - algebra such that all nonzero elements are invertible.

(D is "central over k", $Z(D) = k$)

- How many division alg. are there central over $\mathbb{C}$?

$a \in D \setminus \mathbb{C}$ consider subfield of D containing a & $\mathbb{C}$

alg. ext. of $\mathbb{C}$.

$D = \mathbb{C}$.

What about over $\mathbb{R}$?

$\mathbb{R}$ & $\mathbb{H}$ Hamilton's quaternions.

$$\operatorname{Span}_{\mathbb{R}}\{1, i, j, k\}.$$

What about over $\mathbb{Q}$?

There are infinitely many.

Alber - Brauer - Hasse - Noether

A <u>rep</u> of a finite group G over k (char $k = 0$)

V f.d. v.s. over k

$f : G \to GL(V)$ homomorphism.

Another viewpoint: V is a $k[G]$-module

$\uparrow$

group ring.

$$u_g u_h$$

$$u_{gh}$$

Every rep of G over k is completely reducible: i.e. we can write

$$V = \bigoplus_{i=1}^{m} V_i^{\oplus a_i}$$

V_i's are distinct irreducible reps (this decomposition is unique).

no sub reps.

Example: The regular representation of a group G over k is

$V = k[G]$ have G act on V by left mult.

$$g \cdot \sum_{h \in G} c_h u_h = \sum_{h \in G} c_h u_{gh}$$

$$c_h \in k.$$

Now $$k[G] = \bigoplus_{i=1}^{m} V_i^{\oplus a_i}$$

Each V_i is an irreducible representation of $k[G]$. So V_i is going to be closed under mult.

$$x, y \in V_i$$

$$x \in k[G]$$

$$xy = \underset{\substack{\uparrow \\ k[G]}}{x} \cdot \underset{\substack{\uparrow \\ \text{inside of a } k[G]\text{-module}}}{y} \in V_i$$

$k[G]$ is a semisimple left $k[G]$-module and each V_i is a simple left ideal

A ring R is semisimple if

- it is a semisimple R-module
- or a sum of two-sided simple ideals.

Let A_i be the sum of left ideals isomorphic to V_i:

$$A_i = V_i \, k[G] = \sum V \text{ that are iso to } V_i$$

Then A_i is a twosided simple ideal

$$k[G] = \bigoplus_{i=1}^{m} \underbrace{A_i}_{?}$$

$$\underbrace{V_i^{\oplus a_i}}$$

Theorem (Wedderburn)

A f.d. simple algebra over k.

$\iff A \cong M_n(D)$ for some n and division alg D over k.

$(\Leftarrow)$ for the same reason $M_n(k)$ is

What are the simple left ideals in $M_n(D)$?

$I_r = \{ M \mid$ the only nonzero entries of M are in the rth column $\}$

$$M_n(D) = \bigoplus_r I_r$$

Given A the minimal left ideals L_i of A can be formed into these "column spaces" over the division algebra

$$\underline{\underline{D}} = \underline{\underline{\mathrm{End}_A(L)}}$$

$$k[G] = \bigoplus_{i=1}^{m} A_i \cong \bigoplus_{i=1}^{m} M_{n_i}(D_i)$$

where D_i are division algebras not necessarily central over k,

<u>Main Question</u>: Which Division algebras show up in this decomp?

$k = \mathbb{C}$

$$\mathbb{C}[G] = \bigoplus_{i=1}^{m} V_i^{\oplus a_i}$$

$$= \bigoplus M_{n_i}(\mathbb{C})$$

$$V_i^{\oplus a_i} \cong A_i \cong M_{a_i}(\mathbb{C})$$

$$\dim_{\mathbb{C}} V_i = a_i$$

What happens over $\mathbb{Q}$?

1) $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$ $\quad G = \{1, \sigma\}$

$V_1 = \text{span}\{1+\sigma\}$ is the trivial rep

$$1 \cdot (1+\sigma) = 1+\sigma$$
$$\sigma \cdot (1+\sigma) = \sigma + 1 = 1+\sigma$$

$G \curvearrowright V_1$ is trivial

i.e. "trivial rep"

$V_2 = \text{span}\{1-\sigma\}$

$$1 \cdot (1-\sigma) = 1-\sigma$$
$$\sigma(1-\sigma) = \sigma - 1 = -(1-\sigma)$$

"alternating rep"

2) $\mathbb{Q}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{Q}[x]/(x^3-1)$

$$\cong \underbrace{\mathbb{Q}[x]/(x-1)}_{\cong \mathbb{Q}} \times \underbrace{\mathbb{Q}[x]/(x^2+x+1)}_{\cong L}$$

$V_1 = \text{span}\{1+\sigma+\sigma^2\}$

$V_2 = \text{span}\{1-\sigma, \sigma-\sigma^2\}$

$$M_1(\mathbb{Q}) \oplus M_1(L)$$

3) $\mathbb{Q}[Q_8]$ $\qquad Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$$\cong \underbrace{\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}}_{4\ 1\text{-dim'l reps}} \oplus \underbrace{(-1,-1)_{\mathbb{Q}}}_{M_1(D)}$$

4 dim'l